

ECON-33200: Theory of Income III

New Keynesian Model

Siddarth Venkatesh

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In this note, I try to combine the ideas present in the lecture notes by Guido Lorenzoni for the Theory of Income III course and the formulations in Jordi Gali's textbook in a way that I find most intuitive and comprehensive. Apologies for any errors, which are entirely my own.

1 Setup

1.1 Household

An infinitely lived representative consumer solves the problem:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to the period-wise budget constraints:

$$P_t C_t + Q_t B_t \leq W_t N_t + B_{t-1} + T_t$$

where C_t is consumption, N_t is labor supply, P_t is the price of the consumption good, W_t is the nominal wage, B_t is the amount of risk-free nominal bonds purchased at time t that yield one unit at time $t + 1$, and Q_t is the price of the bond at time t . For simplicity, we assume that the utility function is additively separable in consumption and labor, and takes the form:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

where $\sigma > 0$ is the coefficient of relative risk aversion and $1/\varphi$ is the Frisch elasticity of labor supply. We can represent the household's problem either as a sequence problem (Lagrangian) or a recursive functional problem (Bellman equation). I use the Bellman approach in this note. The consumer problem can be written as:

$$\begin{aligned} V_t(B_{t-1}) &= \max_{C_t, N_t, B_t} U(C_t, N_t) + \beta \mathbb{E}_t V_{t+1}(B_t) \\ \text{s.t. } P_t C_t + Q_t B_t &\leq W_t N_t + B_{t-1} + T_t \end{aligned}$$

This yields the first-order conditions:

$$\begin{aligned} U_c(C_t, N_t) &= \lambda P_t \\ U_n(C_t, N_t) &= -\lambda W_t \\ \beta \mathbb{E}_t V'_{t+1}(B_t) &= \lambda Q_t \end{aligned}$$

The envelope condition is:

$$V'_t(B_{t-1}) = \lambda$$

Combined with the first FOC, we get:

$$U_c(C_t, N_t) = V'_t(B_{t-1})P_t$$

Substituting the envelope condition forward into period $t + 1$ and using the third FOC, we get the Euler equation:

$$Q_t = \beta \mathbb{E}_t \left[\frac{U_c(C_{t+1}, N_{t+1})}{U_c(C_t, N_t)} \frac{P_t}{P_{t+1}} \right]$$

Substituting in the functional form of the utility function:

$$Q_t = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] \quad (\text{EE})$$

$$N_t^\varphi C_t^\sigma = \frac{W_t}{P_t} \quad (\text{LS})$$

we have the desired Euler equation (EE) and labor supply (LS) conditions.

1.2 Firms

Firms in this economy are organized in a structure akin to the monopolistic competition model we learned in Theory of Income II. The only difference is that firms face a Calvo price-setting friction, where they face an exogenous probability θ of not being able to adjust their price in any given period. This will change the firm optimization problem from a static to a dynamic one.

1.2.1 Final Good

The final (consumption) good is produced by a competitive firm, which combines a continuum of intermediate goods $Y(i)$, $i \in [0, 1]$ according to a CES aggregator:

$$\begin{aligned} \max_{Y, Y(i)} \quad & P_t Y - \int_0^1 P_t(i) Y(i) di \\ \text{s.t.} \quad & Y = \left(\int_0^1 Y(i)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

This problem yields the familiar FOCs:

$$P_t = \mu$$

$$P_t(i) = \mu \left(\frac{Y(i)}{Y} \right)^{-\frac{1}{\varepsilon}}$$

Together, these imply the usual CES demand function for each intermediate good

$$Y(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y$$

1.2.2 Intermediate Goods

Each intermediate good producer, $i \in [0, 1]$ is a monopolist, whose problem comprises of setting a price, choosing a level of output, and hiring labor to produce that output. Since output and labor are constrained by the demand and production functions, the firm's problem is simply one of choosing to set a price. Depending on the realization of the Calvo shock, however, the firm may not be able choose its price, in which case it does not have a problem to solve. Therefore, we are only interested in the problem of firms that adjust their price at time t .

The period $t + j$ profits of a firm that last set its price at time t are given by:

$$D_{t+j|t} = \frac{P_t(i)}{P_{t+j}} Y_{t+j|t}(i) - \frac{1}{P_{t+j}} \Psi_{t+j}(Y_{t+j|t}(i))$$

where

$$Y_{t+j|t}(i) = \left(\frac{P_t(i)}{P_{t+j}} \right)^{-\varepsilon} Y_{t+j}$$

and its cost function $\Psi_{t+j}(Y_{t+j}(i))$ is given by:

$$\begin{aligned} \Psi_{t+j}(Y_{t+j}(i)) &= W_{t+j} N_{t+j}(i) \\ &\equiv W_{t+j} \left(\frac{Y_{t+j}(i)}{A_{t+j}} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

This implies the marginal cost function:

$$\Psi'_{t+k}(Y_{t+k}(i)) = \frac{1}{(1-\alpha)Y_{t+k}(i)} \Psi_{t+k}(Y_{t+k}(i))$$

Let $\Psi_{t+j|t} = \Psi_{t+j}(Y_{t+j|t}(i))$ and $\Psi'_{t+j|t} = \Psi'_{t+j}(Y_{t+j|t}(i))$. When a firm sets its price at time t , there is a θ^j probability that it will not be able to adjust its price until time $t + j$. Since a firm's previous pricing decisions do not affect its current pricing decision (there are no costs associated with the magnitude of price changes), it is enough to consider realizations of the Calvo shocks where the firm does not adjust its price. The firm maximizes is expected discounted profits,

$$\max_{P_t(i)} \sum_{j=0}^{\infty} \mathbb{E}_t M_{t,t+j} D_{t+j}$$

subject to the period-wise demand constraints:

$$Y_{t+j}(i) = \left(\frac{P_{t+j}(i)}{P_{t+j}} \right)^{-\varepsilon} Y_{t+j}$$

and where

$$M_{t,t+j} \equiv (\beta\theta)^j \left(\frac{C_{t+j}}{C_t} \right)^{-\sigma}$$

is the stochastic discount factor. Given the Calvo structure, $P_t(i)$ features only in those histories where the firm does not adjust its price perpetually. Then, substituting the demand functions into the objective function and solving the unconstrained problem, the optimal price P_t^* (note we impose a symmetric equilibrium here as all the firm problems are identical) satisfies the first-order condition:

$$\begin{aligned} \sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} \left(\frac{Y_{t+j|t}(i)}{P_{t+j}} - \varepsilon \frac{P_t^*}{P_{t+j}} \frac{Y_{t+j|t}(i)}{P_t^*} + \frac{\Psi'_{t+j|t}}{P_{t+j}} \left(-\varepsilon \frac{Y_{t+j|t}(i)}{P_t^*} \right) \right) \right] &= 0 \\ \sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} Y_{t+j|t}(i) \left((1-\varepsilon) \frac{P_t^*}{P_{t+j}} - \varepsilon \frac{\Psi'_{t+j|t}}{P_{t+j}} \right) \right] &= 0 \\ \Rightarrow \sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} Y_{t+j|t}(i) \left(\frac{P_t^*}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} \frac{\Psi'_{t+j|t}}{P_{t+j}} \right) \right] &= 0 \quad (\text{PS}) \end{aligned}$$

This gives us the optimal price-setting (PS) equation. The PS equation is a forward-looking condition that can be interpreted as a weighted average of the firm's future deviations from the flexible price optimum. Note that when $\theta = 0$, i.e. the flexible price case, the PS equation reduces to setting $P_t^* = \mathcal{M} \Psi'_t(Y_t(i))$, where $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$ is the optimal flexible price markup. When prices are sticky, Calvo shocks that prevent the firm from adjusting its price in the future can cause the firm to deviate from this optimal markup, so its optimal price setting decision captures a notion of setting (weighted) average deviations from the flexible price optimum to zero.

1.2.3 Pricing

From the final good firm's problem, we have the price index:

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

We wish to track its evolution over time, which is what we measure as inflation. Let $S(t)$ be the set of firms that do not adjust their price at time t . Then, we can rewrite the price index as:

$$\begin{aligned} P_t &= \left(\int_{S(t)} P_{t-1}(i)^{1-\varepsilon} di + \int_{S^c(t)} (P_t^*)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \\ &= (\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon})^{\frac{1}{1-\varepsilon}} \end{aligned}$$

Since the Calvo shock is memoryless and i.i.d across firms, the set of firms that do not adjust their price at time t have the same price distribution as the set of all firms. Hence, the integral over $S(t)$ is simply the measure of firms that do not adjust their price (θ) times $P_{t-1}^{1-\varepsilon}$. This gives us the inflation dynamics:

$$\Pi_t^{1-\varepsilon} = \theta + (1 - \theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon}$$

1.3 Market Clearing

An equilibrium in this economy requires that the labor market, goods market, and bond markets clear. These conditions are, together, given by:

$$\begin{aligned} N_t &= \int_0^1 N_t(i) di \\ Y_t &= C_t \\ B_t &= 0 \end{aligned}$$

From the production function of intermediate good firms:

$$\begin{aligned} N_t &= \int_0^1 \left(\frac{Y_t(i)}{A_t} \right)^{\frac{1}{1-\alpha}} di \\ \implies A_t N_t^{1-\alpha} &= \left(\int_0^1 Y_t(i)^{\frac{1}{1-\alpha}} di \right)^{1-\alpha} \end{aligned}$$

Substituting in the demand function for intermediate goods:

$$\begin{aligned} A_t N_t^{1-\alpha} &= \left(\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} Y_t^{\frac{1}{1-\alpha}} di \right)^{1-\alpha} \\ &= Y_t \left(\int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di \right)^{1-\alpha} \end{aligned}$$

Defining the price dispersion index:

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} di$$

we can use the above expression and the goods market clearing condition to get:

$$A_t N_t^{1-\alpha} = C_t \Delta_t^{1-\alpha}$$

Furthermore, note that the price dispersion amongst firms that do not adjust their price at time t is the same as the price dispersion among all firms at time $t - 1$. Hence, the price dispersion index evolves according to:

$$\Delta_t = \theta \Delta_{t-1} \left(\frac{P_{t-1}}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + (1 - \theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}}$$

The bond market clearing condition in this simple economy is trivial, since households can only borrow privately and there is no government debt. However, this can be relaxed by introducing a government that issues bonds to finance its expenditures. So long as the bonds are nominal, and there are no physical assets present, the goods market clearing condition and its relationship to the labor market clearing condition and price dispersion index will remain unchanged.

2 Equilibrium

2.1 Closing the model

We need two additional elements to close the model. First, we need to specify a stochastic process for the technology shock A_t . For instance, A_t can be determined by an AR(1) process:

$$a_t = \rho_a a_{t-1} + \epsilon_t$$

where ϵ_t is a white noise process and $a_t = \log A_t$. Second, we need to specify a monetary policy rule that determines the nominal interest rate i_t as a function of the state of the economy. An example is where the central bank follows a Taylor rule:

$$i_t = i^* + \phi_\pi \pi_t$$

where $i_t \approx -\log Q_t$. Letting $i^* = \rho \equiv -\log \beta$, bond prices $Q_t = \frac{1}{1+i_t}$ are determined by:

$$1 + i_t = \beta^{-1} \Pi_t^{\phi_\pi}$$

2.2 Stationary equilibrium

Assume A_t follows a Markov process. A stationary equilibrium of the model is a collection of stochastic processes $\left\{C_t, N_t, B_t, Y_t, P_t, W_t, Q_t, \{P_t(i), N_t(i), Y_t(i)\}_{i \in [0,1]}\right\}_{t=0}^{\infty}$ such that the following conditions hold:

- These stochastic processes are measurable with respect to A_t .
- They solve the household problem in each period t .
- They solve the final good firm's problem in each period t .
- They solve the intermediate good firm's problem in every period where the firm adjusts its price
- All markets clear in every period

We wish to characterize the equilibrium as a system of equations, and determine the existence of a stationary equilibrium that satisfies these conditions. These equations include the household

optimality conditions:

$$Q_t = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right]$$

$$N_t^\varphi C_t^\sigma = \frac{W_t}{P_t}$$

The intermediate good firm's optimality condition, i.e. the demand function for each intermediate good i :

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t$$

The optimal-price setting condition for every firm that adjusts its price at time t :

$$\sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} Y_{t+j}(i) \left(\frac{P_t^*}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} \frac{\Psi'_{t+j}(Y_{t+j}(i))}{P_{t+j}} \right) \right] = 0$$

Market clearing conditions:

$$N_t = \int_0^1 N_t(i) di$$

$$Y_t = C_t$$

$$B_t = 0$$

$$A_t N_t^{1-\alpha} = C_t \Delta_t^{1-\alpha}$$

and pricing dynamics:

$$\Pi_t^{1-\varepsilon} = \theta + (1 - \theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon}$$

$$\Delta_t = \theta \Delta_{t-1} \left(\frac{P_{t-1}}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}} + (1 - \theta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{\varepsilon}{1-\alpha}}$$

Note that the condition for the demand function for good i can be equivalently represented by the following condition on marginal costs faced by firms in period $t + j$ when the last set their price at time t :

$$\Psi'_{t+j|t} = \frac{W_{t+j}}{(1 - \alpha)A_{t+j}} \left(\frac{Y_{t+j}}{A_{t+j}} \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{P_t(i)}{P_{t+j}} \right)^{-\frac{\varepsilon\alpha}{1-\alpha}}$$

2.3 Constructing a zero-inflation equilibrium

To have an equilibrium with zero inflation, we need to have $P_t^* = P_{t-1} = P_t$ for all t . Suppose w.l.o.g. that $P_t = 1$. From the evolution of price dispersion, we get:

$$\Delta_t = \theta \Delta_{t-1} + (1 - \theta)$$

Note that for the equilibrium to be stationary, it must be that $\Delta_t = 1$ in all periods, which allows aggregate allocations to be independent of the time index. The price setting equation reduces to:

$$\sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} Y_{t+j}(i) \left(1 - \frac{\varepsilon}{\varepsilon - 1} \Psi'_{t+j}(Y_{t+j|t}(i)) \right) \right] = 0$$

One solution to this equation is:

$$\Psi'_{t+j|t} = \frac{\varepsilon - 1}{\varepsilon}$$

for all j . This is equivalent to:

$$\frac{\varepsilon - 1}{\varepsilon} = \frac{W_{t+j}}{(1 - \alpha)A_{t+j}} \left(\frac{Y_{t+j}}{A_{t+j}} \right)^{-\frac{\alpha}{1-\alpha}}$$

Together with goods market clearing and the labor supply conditions:

$$\begin{aligned} A_t N_t^{1-\alpha} &= Y_t \\ N_t^\varphi Y_t^\sigma &= W_t \end{aligned}$$

we have three equations in three unknowns, Y_t , N_t , and W_t , which yield solutions in terms of only the contemporaneous technology shock A_t . For this to be an equilibrium, we also need the Euler equation to be satisfied. This gives us:

$$Q_t = \beta \mathbb{E}_t \left[\left(\frac{Y(A_{t+1})}{Y(A_t)} \right)^{-\sigma} \right]$$

Since A_t follows a Markov process, the right-hand side is a function of A_t alone, and we can let $Q_t = Q(A_t)$. This gives us a complete characterization of a zero-inflation equilibrium.

The divine coincidence result here is that there exists an optimally chosen $i(A_t)$ that can stabilize inflation at zero, which also, in turn, stabilizes the output gap. We show, using the linearized model, that a stringent inflation targeting rule can also achieve this result, under certain conditions, without requiring an explicit state-contingent policy rule. By controlling inflation, the central bank controls the output gap.

2.4 Non-stochastic steady state

When computing the response of the economy to shocks, we linearize the model around a non-stochastic steady state. For small shocks, the first order approximation captures the dynamics of the economy well. A non-stochastic steady state is one where there is no uncertainty, i.e. $A_t = A$ for all t . A non-stochastic steady state with no inflation would simply be a special case of the zero-inflation stationary equilibrium, with no uncertainty in the technology process. On the other hand, we can also construct a steady state with positive trend inflation.

In this case, prices will not have a steady state value. However, we can capture stationarity in the ratio of the price decision to either past or current prices. To that end, let $X_t = P_t^*/P_t$.

From the price dynamics equation,

$$1 = \theta \Pi_t^{\varepsilon-1} + (1 - \theta) X_t^{1-\varepsilon}$$

where the steady state inflation Π determines the steady state X . Price dispersion will also have a steady state solution. Though price dispersion need not be initialized at this steady state value, it is necessary for other aggregate variables to remain at the steady state. The steady state price dispersion index solves:

$$\Delta = \theta \Delta \Pi^{\frac{\varepsilon}{1-\alpha}} + (1 - \theta) X^{-\frac{\varepsilon}{1-\alpha}}$$

The firm's price setting condition, the marginal cost function, labor supply condition, and the goods market clearing condition together give us a system of equations that determine steady state output, labor, real wage, and marginal cost:

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty} (\beta\theta)^j \Pi^{j\varepsilon} \left(X \Pi^{-j} - \mathcal{M}MC \Pi^{\frac{\varepsilon\alpha}{1-\alpha}} \right) \\ MC &= W(1 - \alpha)^{-1} A^{-\frac{1}{1-\alpha}} X^{-\frac{\varepsilon\alpha}{1-\alpha}} Y^{\frac{\alpha}{1-\alpha}} \\ N^\varphi Y^\sigma &= W \\ AN^{1-\alpha} &= Y \Delta^{1-\alpha} \end{aligned}$$

Note that steady state marginal cost here refers to the marginal cost faced by a firm that adjusts its price. Along periods of non-adjustment, its marginal cost deviates from this value to a power of the rate of inflation. Further, note that W in this case refers to a steady state real wage. Since prices are growing at a constant rate, the nominal wage must also grow at the same rate to keep the real wage constant. Finally, we need the Euler equation to be satisfied for this steady state to be an equilibrium. This gives us the value of the steady state nominal rate rate:

$$1 + i = \frac{\Pi}{\beta}$$

The implication of this equation is that the gross real interest rate in the steady state is still equal to the inverse of the discount factor, even when there is positive trend inflation. This gives us a complete system that we can now linearize around to compute the response of the economy to shocks.

3 Linearization

3.1 Dynamic IS Equation

Recall the Euler equation:

$$1 = \beta(1 + i_t) \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right]$$

Around the non-stochastic steady state with $C_t = C_{t+1} = \bar{C}$, $\Pi_{t+1} = \Pi$, and $1 + i_t = 1 + i = \Pi\beta^{-1}$,

$$1 = e^{\iota_t} \mathbb{E}_t [\exp(-\sigma(c_{t+1} - c_t) - (\pi_{t+1} - \pi))]$$

A first order approximation yields:

$$0 \approx \iota_t + \mathbb{E}_t [-\sigma(c_{t+1} - c_t) - (\pi_{t+1} - \pi)]$$

The linearized Euler equation can then be written as:

$$c_t = \mathbb{E}_t \left[c_{t+1} - \frac{1}{\sigma} (\iota_t - \pi_{t+1} + \pi) \right]$$

Since we defined $i = \Pi\beta^{-1} - 1 \approx \pi + \rho$, and $\iota_t \approx i_t - i$, we can substitute in the goods market clearing condition, $y_t = c_t$, to get the dynamic IS equation:

$$y_t = \mathbb{E}_t \left[y_{t+1} - \frac{1}{\sigma} (i_t - \pi_{t+1} - \rho) \right] \quad (\text{IS})$$

Alternatively, letting $\hat{i}_t = \beta(1 + i_t) - 1 \approx i_t - \rho$, we get the version in the lecture notes:

$$y_t = \mathbb{E}_t \left[y_{t+1} - \frac{1}{\sigma} (\hat{i}_t - \pi_{t+1}) \right]$$

We can also log-linearize the labor supply condition and the aggregate resource constraint:

$$\begin{aligned} w_t - p_t &= \varphi n_t + \sigma c_t \\ (1 - \alpha)d_t + y_t &= a_t + (1 - \alpha)n_t \end{aligned}$$

where $d_t \equiv \log(\Delta_t/\Delta)$. When inflation is 0 in the steady state, the aggregate resource constraint reduces to the versions in the lecture notes.

3.2 New Keynesian Phillips Curve

We can linearize inflation dynamics around the steady state to get:

$$\pi_t = \pi + \frac{\Pi^{1-\varepsilon} - \theta}{\theta} (p_t^* - p_t) \quad (1)$$

When $\pi = 0$ in the steady state, this reduces to $\pi_t = \frac{1-\theta}{\theta} (p_t^* - p_t)$. Using this equation, we can derive the NKPC by linearizing the PS equation around the steady state. In the case of zero steady state inflation, we can obtain a simple one-step forward looking expression for the NKPC. We first go through this case. First, we rewrite the PS equation as:

$$\sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+j}} Y_{t+j|t}(i) \left(\frac{P_t^*}{P_t} - \mathcal{M} M C_{t+j|t} \frac{P_{t+j}}{P_t} \right) \right] = 0$$

At the non-stochastic steady state, real marginal cost equals the inverse of the markup in every period, and $X = 1$. This means, we can ignore first order contributions from the rest of the

equation. Then, the linearized PS equation is:¹

$$\sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t [p_t^* - p_t - mc_{t+j|t} - (p_{t+j} - p_t)] = 0 \quad (2)$$

Rearranging this equation and substituting (1) gives:

$$\pi_t = \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t [mc_{t+j|t} + (p_{t+j} - p_t)]$$

To obtain the NKPC, we need to rewrite marginal cost deviations in terms of averages, rather than deviations specific to firms adjusting their price at time t . Towards that end, use the expression for $\Psi'_{t+j|t}$ to get:

$$\begin{aligned} mc_{t+j|t} &= mc_{t+j} - \frac{\varepsilon\alpha}{1 - \alpha} (p_t^* - p_{t+j}) \\ &= mc_{t+j} - \frac{\varepsilon\alpha}{1 - \alpha} \frac{\theta}{1 - \theta} \pi_t + \frac{\varepsilon\alpha}{1 - \alpha} (p_{t+j} - p_t) \end{aligned}$$

Substituting this into (2) and rearranging gives:

$$\pi_t = \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \sum_{j=0}^{\infty} (\beta\theta)^j \mathbb{E}_t \left[\frac{1 - \alpha}{1 - \alpha + \varepsilon\alpha} mc_{t+j} + (p_{t+j} - p_t) \right]$$

Using the law of iterated expectations, splitting the sum, and adding and subtracting p_{t+1} gives:

$$\pi_t = \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \frac{1 - \alpha}{1 - \alpha + \varepsilon\alpha} mc_{t+j} + \beta \mathbb{E}_t [\pi_{t+1}]$$

To rewrite this in terms of the output gap, we can use the resource constraint and labor supply equations. First, define the flexible price level of output, y_{t+j}^* , as the level of output that solves $mc_{t+j} = 0$. Recall the marginal cost deviation:

$$mc_{t+j} = w_{t+j} - p_{t+j} - \frac{1}{1 - \alpha} (a_{t+j} - \alpha y_{t+j})$$

From the labor supply condition and resource constraint, we have:

$$\begin{aligned} w_{t+j} - p_{t+j} &= \varphi n_{t+j} + \sigma y_{t+j} \\ &= \frac{\varphi}{1 - \alpha} (y_{t+j} - a_{t+j}) + \sigma y_{t+j} \end{aligned}$$

Combining these two equations and rewriting in terms of deviations from the flexible price outcome gives:

$$mc_{t+j} = \left(\sigma + \frac{\varphi - \alpha}{1 - \alpha} \right) \underbrace{(y_{t+j} - y_{t+j}^*)}_{\equiv \hat{y}_{t+j}}$$

¹There are two ways of going about this. One is to write every variable $Z_t = Ze^{z_t} \approx Z(1 + z_t)$ and eliminate all the steady state terms. Another way is to recall that when approximating around 1, $\log Z_t \approx Z_t - 1$. Then, adding and subtracting 1 gives the desired result.

Defining

$$\kappa \equiv \left(\sigma + \frac{\varphi - \alpha}{1 - \alpha} \right) \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \frac{1 - \alpha}{1 - \alpha + \varepsilon\alpha}$$

we have the NKPC:

$$\pi_t = \beta \mathbb{E}_t [\pi_{t+1}] + \kappa \hat{y}_t \quad (\text{NKPC})$$

(*Optional Stuff*) In the generalized setting with non-zero inflation, price level dynamics are determined by a system of equations. This system is of the form:

$$\begin{aligned} \pi_t &= (1 - \nu_\pi)\pi + \nu_\pi \mathbb{E}_t [\pi_{t+1}] + \nu_y \hat{y}_t + \nu_\xi \xi_t + \nu_d d_t \\ \xi_t &= \delta_\Gamma \mathbb{E}_t [-\sigma(y_{t+1} - y_t) + (\varepsilon - 1)\pi_{t+1} + \xi_{t+1}] \\ d_t &= \eta_d d_{t-1} + \eta_\pi (\pi_t - \pi) \end{aligned}$$

The extra equations appear because under the zero-inflation steady state, the price dispersion index term does not appear first order. This is no longer true with non-zero trend inflation. A way to generalize the exact form from the zero-inflation case would be to index price changes to the trend. The approach used to derive the coefficients here would be the same as before, just more tedious. This does not come up in class, though.